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CRANK–NICOLSON SCHEME FOR TWO-DIMENSIONAL IN SPACE FRACTIONAL DIFFUSION EQUATIONS WITH FUNCTIONAL DELAY

A two-dimensional in space fractional diffusion equation with functional delay of a general form is considered. For this problem, the Crank–Nicolson method is constructed, based on shifted Grunwald–Letnikov formulas for approximating fractional derivatives with respect to each spatial variable and using piecewise linear interpolation of discrete history with continuation extrapolation to take into account the delay effect. The Douglas scheme is used to reduce the emerging high-dimensional system to tridiagonal systems. The residual of the method is investigated. To obtain the order of the method, we reduce the systems to constructions of the general difference scheme with heredity. A theorem on the second order of convergence of the method in time and space steps is proved. The results of numerical experiments are presented.

Keywords: diffusion equation, two spatial coordinates, functional delay, Grunwald–Letnikov approximation, Crank–Nicolson method, factorization, order of convergence.

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Introduction

The effect of functional delay endows mathematical models with essential features and the most effective methods for studying such models are numerical methods. For partial differential equations, grid methods for solving equations with functional delay have been sufficiently developed, see [1]. Numerical methods have also been developed for diffusion equations that are fractional in one-dimensional space, see, for example, [2, 3], including those with the effect of functional delay [4]. For two or more spatial dimensions with fractional space derivatives of diffusion-type equations, there are a large number of works devoted mainly to methods for solving the arising large-dimensional system [5–13], but without the delay effect. In this paper, the results of [4] for equations with functional delay are generalized to the two-dimensional case. The algorithm from the article [6] is taken as a basis. Note that earlier this problem was considered [16], where a method of the first order in time and space steps was constructed. In this paper, a method of the second order in time and space steps is constructed and proof of the order of convergence is given.

§ 1. Formulation of the problem

Consider an equation of the form

$$\begin{aligned}\frac{\partial u(x,y,t)}{\partial t} &= d_1^{+}\frac{\partial_{+}^{\alpha_1}u(x,y,t)}{\partial x} + d_1^{-}\frac{\partial_{-}^{\alpha_1}u(x,y,t)}{\partial x} + d_2^{+}\frac{\partial_{+}^{\alpha_2}u(x,y,t)}{\partial y} + \\ &+ d_2^{-}\frac{\partial_{-}^{\alpha_2}u(x,y,t)}{\partial y} + f(x,y,t,u(x,y,t),u_t(x,y,\cdot)),\ (x,y,t)\in\Omega\times(0,T],\end{aligned} \tag{1.1}$$

where $0 \leqslant t \leqslant T$, $a_1 \leqslant x \leqslant b_1$, $a_2 \leqslant y \leqslant b_2$ are independent variables, $u(x,y,t)$ is the required function, $u_t(x,y,\cdot) = \{u(x,y,t+s), \tau \leqslant s < 0\}$ is prehistory of the required function by the time t, $\tau > 0$ is the value of delay, coefficients d_i^{+}, d_i^{-}, $i = 1, 2$ are positive, $\Omega = (a_1,b_1)\times(a_2,b_2)$.

The boundary conditions are given

$$u(x,y,t) = 0,\ (x,y,t)\in\partial\Omega\times(0,T], \tag{1.2}$$

and the initial conditions are given

$$u(x,y,t)=\varphi(x,y,t),\ (x,y,t)\in\hat{\Omega}\times[-\tau,0], \tag{1.3}$$

where $\partial\Omega$ is the set Ω boundary, $\hat{\Omega}$ is the closure of the set Ω.

Left-sided and right-sided Riemann–Liouville fractional derivatives of orders α_i, $1<\alpha_i<2$, $i=1,2$ are defined by the formulas

$$\frac{\partial_+^{\alpha_1}u(x,y,t)}{\partial x}=\frac{1}{\Gamma(2-\alpha_1)}\frac{d^2}{dx^2}\int\limits_{a_1}^{x}\frac{u(\xi,y,t)}{(x-\xi)^{\alpha_1-1}}d\xi, \tag{1.4}$$

$$\frac{\partial_-^{\alpha_1}u(x,y,t)}{\partial x}=\frac{1}{\Gamma(2-\alpha_1)}\frac{d^2}{dx^2}\int\limits_{x}^{b_1}\frac{u(\xi,y,t)}{(x-\xi)^{\alpha_1-1}}d\xi, \tag{1.5}$$

$$\frac{\partial_+^{\alpha_2}u(x,y,t)}{\partial y}=\frac{1}{\Gamma(2-\alpha_2)}\frac{d^2}{dy^2}\int\limits_{a_2}^{y}\frac{u(x,\xi,t)}{(y-\xi)^{\alpha_2-1}}d\xi, \tag{1.6}$$

$$\frac{\partial_-^{\alpha_2}u(x,y,t)}{\partial y}=\frac{1}{\Gamma(2-\alpha_2)}\frac{d^2}{dy^2}\int\limits_{y}^{b_2}\frac{u(x,\xi,t)}{(y-\xi)^{\alpha_2-1}}d\xi. \tag{1.7}$$

We assume that the solution $u(x,y,t)$ to the problem (1.1)–(1.3) exists and is unique. Moreover, in proving the convergence of numerical algorithms, we will assume the necessary smoothness of the solution $u(x,y,t)$.

We denote by $Q=Q[-\tau,0)$ the set of functions $v(s)$, piecewise continuous on $[-\tau,0)$ with the finite number of discontinuity points of the first kind, at the discontinuity points that are continuous on the right. We define the norm of functions on Q by the ratio

$$\|v(\cdot)\|_Q=\sup_{s\in[-\tau,0)}|v(s)|.$$

Additionally, we will assume that the functional $f(x,y,t,u,v(\cdot))$ is defined on $\Omega\times[0,T]\times R\times Q$ and Lipschitz in the last two arguments, that is, there is a constant L_f, such that for all $(x,y)\in\Omega$, $t\in[0,T]$, $u^1\in R$, $u^2\in R$, $v^1(\cdot)\in Q$, $v^2(\cdot)\in Q$ the following inequality is satisfied:

$$|f(x,y,t,u^1,v^1(\cdot))-f(x,y,t,u^2,v^2(\cdot))|\leqslant L_f(|u^1-u^2|+\|v^1(\cdot)-v^2(\cdot)\|_Q). \tag{1.8}$$

§ 2. Difference scheme

§ 2.1. Discretization. Interpolation

Introduce the time step $\Delta=\frac{\tau}{M_0}$, where M_0 is a natural number and let $M=[\frac{T}{\Delta}]$. Introduce the points $t_k=k\Delta$, $k=-M_0,\ldots,M$. Let's also introduce the points $\bar{t}_k=t_k+\Delta/2$, $k=0,\ldots,M-1$.

Let's split the segment $[a_1,b_1]$ into parts with a step $h_1=(b_1-a_1)/N_1$, and let $x_i=a_1+ih_1$, $i=0,\ldots,N_1$. Divide the segment $[a_2,b_2]$ into parts with a step $h_2=(b_2-a_2)/N_2$, and let $y_j=a_2+jh_2$, $j=0,\ldots,N_2$.

The approximation of the function $u(x_i,y_j,t_k)$ at the grid nodes will be denoted by $u_{i,j}^k$.

For any fixed $i=0,\ldots,N_1$ and $j=0,\ldots,N_2$ we introduce a discrete prehistory at the time t_m, $m=0,\ldots,M$: $\{u_{i,j}^k\}_m=\{u_{i,j}^k, m-M_0\leqslant j\leqslant m\}$. By the interpolation operator (with extrapolation) of discrete prehistory we mean the mapping I: assigning the discrete prehistory $\{u_{i,j}^k\}_m$ to the function $u^I(t)=u_{i,j}^m(t)$ defined on $[t_m-\tau,\bar{t}_m]$.

We say that the interpolation operator has the order of error p on the exact solution, if there are constants C_1 and C_2 such that for all i, j, m and $t \in [t_m - \tau, \bar{t}_m]$ the following inequality holds:

$$|u^m_{i,j}(t) - u(x_i, y_j, t)| \leqslant C_1 \max_{m-M_0 \leqslant k \leqslant m} |u^k_{i,j} - u(x_i, y_j, t_k)| + C_2 \Delta^p.$$

In what follows, for the methods under consideration, we will use piecewise linear interpolation

$$u^I(t) = u^m_{i,j}(t) = u^{k-1}_{i,j} \frac{t_k - t}{\Delta} + u^k_{i,j} \frac{t - t_{k-1}}{\Delta}, \; t_{k-1} \leqslant t < t_k, \; k \leqslant m. \tag{2.1}$$

with extrapolation by continuation

$$u^I(t) = u^m_{i,j}(t) = u^{m-1}_{i,j} \frac{t_m - t}{\Delta} + u^m_{i,j} \frac{t - t_{m-1}}{\Delta}, \; t_m \leqslant t \leqslant \bar{t}_m. \tag{2.2}$$

This interpolation operator is of second order if the exact solution is twice continuously differentiable with respect to t [14, p. 98, 102].

In addition, this operator is Lipschitz with the constant $L_I = 2$ in the following sense. Let

$$v^I(t) = v^m_{i,j}(t) = v^{k-1}_{i,j} \frac{t_k - t}{\Delta} + v^k_{i,j} \frac{t - t_{k-1}}{\Delta}, \; t_{k-1} \leqslant t < t_k, \; k \leqslant m,$$

$$v^I(t) = v^m_{i,j}(t) = v^{m-1}_{i,j} \frac{t_m - t}{\Delta} + v^m_{i,j} \frac{t - t_{m-1}}{\Delta}, \; t_m \leqslant t \leqslant \bar{t}_m,$$

then, for any $t \in [t_m - \tau, \bar{t}_m]$

$$|u^I(t) - v^I(t)| \leqslant L_I \max_{m-M_0 \leqslant k \leqslant m} |u^k_{i,j} - v^k_{i,j}|. \tag{2.3}$$

§ 2.2. Approximation of fractional derivatives

Various algorithms are used to approximate fractional Riemann-Liouville derivatives in grid methods for solving space-fractional diffusion equations [5–8]. Let's choose a method from [6].

Let

$$g^{(\alpha)}_0 = 1, \; g^{(\alpha)}_{k+1} = \big(1 - \frac{\alpha+1}{k+1}\big) g^{(\alpha)}_k, \; \alpha \in (1, 2). \; k = 0, 1, 2, \ldots.$$

We introduce shifted difference operators that approximate the Riemann–Liouville derivatives (1.4)–(1.7), respectively

$${}_{+}D^{\alpha_1}_1 u^m_{i,j} = \frac{1}{h_1^{\alpha_1}} \sum_{k=0}^{i} w^{\alpha}_k u^m_{i-k+1,j}, \quad {}_{-}D^{\alpha_1}_1 u^m_{i,j} = \frac{1}{h_1^{\alpha_1}} \sum_{k=0}^{N_1-i+1} w^{\alpha}_k u^m_{i+k-1,j},$$

$${}_{+}D^{\alpha_2}_2 u^m_{i,j} = \frac{1}{h_2^{\alpha_2}} \sum_{k=0}^{j} w^{\alpha}_k u^m_{i,j-k+1}, \quad {}_{-}D^{\alpha_2}_2 u^m_{i,j} = \frac{1}{h_2^{\alpha_2}} \sum_{k=0}^{N_2-j+1} w^{\alpha}_k u^m_{i,j+k-1}.$$

Here the coefficients w^{γ}_k are determined by the relations

$$w^{\gamma}_0 = \frac{\gamma}{2} g^{\gamma}_0, \quad w^{\gamma}_k = \frac{\gamma}{2} g^{\gamma}_k + \frac{2-\gamma}{2} g^{\gamma}_{k-1}, \quad k \geqslant 1.$$

From the results of [6, formula (2.16)] it follows that if the exact solution $u(x, y, t)$ is four times continuously differentiable with respect to x and y, then

$$\frac{\partial^{\alpha_1}_{+} u(x_i, y_j, t_m)}{\partial x} = {}_{+}D^{\alpha_1}_1 u(x_i, y_j, t_m) + O(h_1^2), \tag{2.4}$$

$$\frac{\partial^{\alpha_1}_{-} u(x_i, y_j, t_m)}{\partial x} = {}_{-}D^{\alpha_1}_1 u(x_i, y_j, t_m) + O(h_1^2), \tag{2.5}$$

$$\frac{\partial^{\alpha_2}_{+} u(x_i, y_j, t_m)}{\partial y} = {}_{+}D^{\alpha_2}_2 u(x_i, y_j, t_m) + O(h_2^2), \tag{2.6}$$

$$\frac{\partial^{\alpha_2}_{-} u(x_i, y_j, t_m)}{\partial y} = {}_{-}D^{\alpha_2}_2 u(x_i, y_j, t_m) + O(h_2^2). \tag{2.7}$$

§ 2.3. Crank–Nicolson method

The two-dimensional fractional analog of the Crank-Nicolson method with functional delay can be written as

$$\frac{u_{i,j}^{m+1}-u_{i,j}^{m}}{\Delta}=\big(d_{1+}^{+}D_1^{\alpha_1}+d_{1-}^{-}D_1^{\alpha_1}+d_{2+}^{+}D_2^{\alpha_2}+d_{2-}^{-}D_2^{\alpha_2}\big)\frac{u_{i,j}^{m+1}+u_{i,j}^{m}}{2}+ \\ +f(x_i,y_j,\bar t_m,u^I(\bar t_m),u_{\bar t_m}^I(\cdot)), \tag{2.8}$$

with initial conditions

$$u_{i,j}^k=\varphi(x_i,y_j,t_k),\quad k=-M_0,\dots,0,\quad i=0,\dots,N_1,\quad j=0,\dots,N_2, \tag{2.9}$$

and boundary conditions

$$u_{i,j}^k=0,\quad k=0,\dots,M,\quad (x_i,y_j)\in\partial\Omega, \tag{2.10}$$

where $u^I(t)$ is the result of piecewise linear interpolation (2.1) with extrapolation (2.2).

Let I be the unit operator,

$$\delta_1^{\alpha_1}=d_{1+}^{+}D_1^{\alpha_1}+d_{1-}^{-}D_1^{\alpha_1},\qquad \delta_2^{\alpha_2}=d_{2+}^{+}D_2^{\alpha_2}+d_{2-}^{-}D_2^{\alpha_2}.$$

Let us give similar for unknown $u_{i,j}^{m+1}$ in (2.8):

$$\big(I-\frac{\Delta}{2}\delta_1^{\alpha_1}-\frac{\Delta}{2}\delta_2^{\alpha_2}\big)u_{i,j}^{m+1}=\big(I+\frac{\Delta}{2}\delta_1^{\alpha_1}+\frac{\Delta}{2}\delta_2^{\alpha_2}\big)u_{i,j}^{m}+ \\ +\Delta f(x_i,y_j,\bar t_m,u^I(\bar t_m),u_{\bar t_m}^I(\cdot)). \tag{2.11}$$

The algorithm requires solving at each time step a system of linear equations of dimension $(N_1-1)\times(N_2-1)$, which is difficult for large N_1 and N_2. Therefore, we replace (2.11) with another algorithm, which is reduced to systems of a special structure, convenient for solving:

$$\big(I-\frac{\Delta}{2}\delta_1^{\alpha_1}\big)\big(I-\frac{\Delta}{2}\delta_2^{\alpha_2}\big)u_{i,j}^{m+1}=\big(I+\frac{\Delta}{2}\delta_1^{\alpha_1}\big)\big(I+\frac{\Delta}{2}\delta_2^{\alpha_2}\big)u_{i,j}^{m}+ \\ +\Delta f(x_i,y_j,\bar t_m,u^I(\bar t_m),u_{\bar t_m}^I(\cdot)). \tag{2.12}$$

The algorithm is supplemented with initial conditions (2.9) and boundary conditions (2.10).

§ 2.4. Solution methods for system (2.12)

To find an effective solution of (2.12), you can use various algorithms: Peaseman–Rachford ADI scheme, Douglas ADI scheme, locally one-dimensional scheme, and other methods.

Special algorithms have also been developed for solving the arising systems of linear equations, taking into account the specifics of the homogeneous part systems (2.12) [6, 7].

We will choose *Douglas ADI scheme*:

$$\big(I-\frac{\Delta}{2}\delta_1^{\alpha_1}\big)V_{i,j}^{m}=\big(I+\frac{\Delta}{2}\delta_1^{\alpha_1}+\Delta\delta_2^{\alpha_2}\big)u_{i,j}^{m}+\Delta f(x_i,y_j,\bar t_m,u^I(\bar t_m),u_{\bar t_m}^I(\cdot)),$$

$$\big(I-\frac{\Delta}{2}\delta_2^{\alpha_2}\big)u_{i,j}^{m+1}=V_{i,j}^{m}-\frac{\Delta}{2}\delta_2^{\alpha_2}u_{i,j}^{m}.$$

§ 3. Error analysis

§ 3.1. Residual of method

For $i = 1, \ldots, N_1 - 1$, $j = 1, \ldots, N_2 - 1$, and $m = 1, \ldots, M - 1$ the residual (without interpolation) of method (2.12) is called the grid function

$$
\begin{gathered}
\psi_{i,j}^m = \big(\frac{1}{\Delta} I - \frac{1}{2}\delta_1^{\alpha_1}\big)\big(I - \frac{\Delta}{2}\delta_2^{\alpha_2}\big)u(x_i, y_j, t_{m+1}) - \\
- \big(\frac{1}{\Delta} I + \frac{1}{2}\delta_1^{\alpha_1}\big)\big(I + \frac{\Delta}{2}\delta_2^{\alpha_2}\big)u(x_i, y_j, t_m) - f(x_i, y_j, \bar{t}_m, u(x_i, y_j, \bar{t}_m), u_{\bar{t}_m}(x_i, y_j, \cdot)).
\end{gathered} \tag{3.1}
$$

L e m m a 3.1 (Residual order without interpolation). *If the function is an exact solution to the problem* (1.1)–(1.3) $u(x, y, t)$ *twice continuously differentiable with respect to* t *and four times continuously differentiable with respect to* x *and* y*, then for the residual (without interpolation) of the method* (2.12) *we have*

$$
|\psi_{i,j}^m| \leqslant C(h_1^2 + h_2^2 + \Delta^2), \quad i = 1, \ldots, N_1 - 1, \quad j = 1, \ldots, N_2 - 1, \quad m = 0, \ldots, M - 1.
$$

P r o o f. We expand $u(x_i, y_j, t_m)$ and $u(x_i, y_j, t_{m+1})$ in Taylor's series in a neighborhood of the point $(x_i, y_j, \bar{t}_m)$ and we use the relations (3.1), (2.4)–(2.7), (1.1):

$$
\begin{gathered}
\psi_{i,j}^m = \big(\frac{1}{\Delta} I - \frac{1}{2}\delta_1^{\alpha_1}\big)\big(I - \frac{\Delta}{2}\delta_2^{\alpha_2}\big)u(x_i, y_j, t_{m+1}) - \big(\frac{1}{\Delta} I + \frac{1}{2}\delta_1^{\alpha_1}\big)\big(I + \frac{\Delta}{2}\delta_2^{\alpha_2}\big)u(x_i, y_j, t_m) - \\
- f(x_i, y_j, \bar{t}_m, u(x_i, y_j, \bar{t}_m), u_{\bar{t}_m}(x_i, y_j, \cdot)) = \\
= \frac{u(x_i, y_j, t_{m+1}) - u(x_i, y_j, t_m)}{\Delta} - \big(\delta_1^{\alpha_1} + \delta_2^{\alpha_2}\big)\frac{u(x_i, y_j, t_{m+1}) + u(x_i, y_j, t_m)}{2} + \\
+ \frac{\Delta}{4}\delta_1^{\alpha_1}\delta_2^{\alpha_2}\big(u(x_i, y_j, t_{m+1}) - u(x_i, y_j, t_m)\big) - f(x_i, y_j, \bar{t}_m, u(x_i, y_j, \bar{t}_m), u_{\bar{t}_m}(x_i, y_j, \cdot)) = \\
= \frac{\partial u(x_i, y_j, \bar{t}_m)}{\partial t} - d_1^+ \frac{\partial_+^{\alpha_1} u(x_i, y_j, \bar{t}_m)}{\partial x} - d_1^- \frac{\partial_-^{\alpha_1} u(x_i, y_j, \bar{t}_m)}{\partial x} - \\
- d_2^+ \frac{\partial_+^{\alpha_2} u(x_i, y_j, \bar{t}_m)}{\partial y} - d_2^- \frac{\partial_-^{\alpha_2} u(x_i, y_j, \bar{t}_m)}{\partial y} + O(h_1^2 + h_2^2 + \Delta^2) - \\
- f(x_i, y_j, \bar{t}_m, u(x_i, y_j, \bar{t}_m), u_{\bar{t}_m}(x_i, y_j, \cdot)) = O(h_1^2 + h_2^2 + \Delta^2).
\end{gathered}
$$

□

The residual with interpolation of the method (2.12) is called the grid function

$$
\begin{gathered}
\hat{\psi}_{i,j}^m = \big(\frac{1}{\Delta} I - \frac{1}{2}\delta_1^{\alpha_1}\big)\big(I - \frac{\Delta}{2}\delta_2^{\alpha_2}\big)u(x_i, y_j, t_{m+1}) - \\
- \big(\frac{1}{\Delta} I + \frac{1}{2}\delta_1^{\alpha_1}\big)\big(I + \frac{\Delta}{2}\delta_2^{\alpha_2}\big)u(x_i, y_j, t_m) - f(x_i, y_j, \bar{t}_m, \hat{u}^m(x_i, y_j, \bar{t}_m), \hat{u}_{\bar{t}_m}^m(x_i, y_j, \cdot)),
\end{gathered} \tag{3.2}
$$

where $\hat{u}^m(x_i, y_j, t)$ for $t \in [\max\{0, t_m - \tau\}, \bar{t}_m]$ is the result of piecewise linear interpolation (2.1) with extrapolation by continuation (2.2) of the discrete history of the exact solution.

L e m m a 3.2 (The residual order with piecewise linear interpolation). *Under the conditions of the previous lemma, for the residual with piecewise linear interpolation of the method* (2.12) *we have*

$$
|\hat{\psi}_{i,j}^m| \leqslant C(h_1^2 + h_2^2 + \Delta^2), \quad i = 1, \ldots, N_1 - 1, \quad j = 1, \ldots, N_2 - 1, \quad m = 0, \ldots, M - 1.
$$

P r o o f. From the definitions (3.1) and (3.2) we obtain

$$|\hat{\psi}_{i,j}^m| \leqslant |\psi_{i,j}^m| + |f(x_i, y_j, \bar{t}_m, u(x_i, y_j, \bar{t}_m), u_{\bar{t}_m}(x_i, y_j, \cdot)) -$$
$$- f(x_i, y_j, \bar{t}_m, \hat{u}^m(x_i, y_j, \bar{t}_m), \hat{u}_{\bar{t}_m}^m(x_i, y_j, \cdot))|.$$

Taking into account the Lipschitz condition (1.8), Lemma 3.1, and the fact that piecewise-linear interpolation with extension extrapolation is of the second order, from this we obtain

$$|\hat{\psi}_{i,j}^m| \leqslant |\psi_{i,j}^m| + L_f(|u(x_i, y_j, \bar{t}_m) - \hat{u}^m(x_i, y_j, \bar{t}_m)| + \|u_{\bar{t}_m}(x_i, y_j, \cdot)) -$$
$$- \hat{u}^m(x_i, y_j, \bar{t}_m), \hat{u}_{\bar{t}_m}^m(x_i, y_j, \cdot))\|_Q) \leqslant$$
$$\leqslant C(h_1^2 + h_2^2 + \Delta^2) + L_f C_2 \Delta^2.$$

□

We determine the error of the method (2.12) $\varepsilon_{i,j}^m = u(x_i, y_j, t_m) - u_{i,j}^m$, $i = 0, \ldots, N_1$, $j = 0, \ldots, N_2$, $m = 0, \ldots, M$.

We say that the error is of the order of $\Delta^p + h_1^{q_1} + h_2^{q_2}$, if there is a constant C such that for all $i = 0, \ldots, N$, $j = 0, \ldots, N$, $m = 0, \ldots, M$ the inequality $|\varepsilon_{i,j}^m| \leqslant C(\Delta^p + h_1^{q_1} + h_2^{q_2})$ holds.

§ 3.2. Vector form of method

Denote by vector $U^m = (u_{1,1}^m, u_{1,2}^m, \ldots, u_{1,N_1-1}^m, u_{2,1}^m, \ldots, u_{N_1-1,N_2-1}^m)^T$, ^{T} is transpose sign, by matrices S_x^-, S_y^-, S_x^+ and S_y^+, corresponding operators $I - \frac{\Delta}{2}\delta_1^{\alpha_1}$, $I - \frac{\Delta}{2}\delta_2^{\alpha_2}$, $I + \frac{\Delta}{2}\delta_1^{\alpha_1}$ and $I + \frac{\Delta}{2}\delta_2^{\alpha_2}$, by vector $F(\bar{t}_m, U_{\bar{t}_m}^m(\cdot))$ absorbs the forcing term, its coordinates are $f(x_i, y_j, \bar{t}_m, u^I(\bar{t}_m), u_{\bar{t}_m}^I(\cdot))$. Then the system (2.12) can be rewritten as an equation

$$S_x^- S_y^- U^{m+1} = S_x^+ S_y^+ U^m + \Delta F(\bar{t}_m, U_{\bar{t}_m}^m(\cdot)). \tag{3.3}$$

It follows from the results of [6] that the matrices S_x^- and S_y^- are invertible, then the equation (3.3) can be solved, i. e. rewritten explicitly

$$U^{m+1} = (S_y^-)^{-1}(S_x^-)^{-1} S_x^+ S_y^+ U^m + \Delta (S_y^-)^{-1}(S_x^-)^{-1} F(\bar{t}_m, U_{\bar{t}_m}^m(\cdot)). \tag{3.4}$$

Let us rewrite method (3.4) as

$$U^{m+1} = SU^m + \Delta\Phi(t_m, I(\{U^k\}_m), \tag{3.5}$$

where $S = (S_y^-)^{-1}(S_x^-)^{-1} S_x^+ S_y^+$,

$$\Phi(t_m, I(\{U^k\}_m) = (S_y^-)^{-1}(S_x^-)^{-1} F(\bar{t}_m, U_{\bar{t}_m}^m(\cdot)), \tag{3.6}$$

$\{U^k\}_m$ is history of vector discrete model, $I(\{U^k\}_m)$ is the result of the action of the operator of piecewise linear interpolation with extrapolation by continuation at the moment t_m.

Let us investigate the order of error of the method (2.12) (in form (3.5)) using the embedding in the general scheme of systems with aftereffect [1, 4].

We define the norm of vector U^m by the relation

$$\|U^m\| = \max_{i,j} |u_{i,j}^m|.$$

The method (3.5) is called stable if there is such constant $\hat{S}$, that

$$\|S^n\| \leqslant \hat{S}$$

for any natural degree n. Here the subordinate norm of the matrix is considered.

The stability of method (2.12) or equivalent to method (3.5) follows from the results [6, Theorem 4.6, Lemma 4.6, Theorem 4.7].

The vector residual (with interpolation) of the method (3.5) is the quantity

$$\nu_m = (Z^{m+1} - SZ^m)/\Delta - \Phi(t_m, I(\{Z^k\}_m)), \quad m = 0, \ldots, M-1,$$

where $Z^m = (u(x_1, y_1, t_m), \ldots, u(x_1, y_{N_1-1}, t_m), u(x_2, y_1, t_m), \ldots, u(x_{N_1-1}, y_{N_2-1}, t_m))^T$ is vector of exact values.

It follows from the result of Lemma 3.2 that under the smoothness conditions formulated in Lemma 3.1, the vector residual has order $h_1^2 + h_2^2 + \Delta^2$, i.e. for any $m = 0, \ldots, M-1$,

$$\|\nu_m\| \leqslant C(h_1^2 + h_2^2 + \Delta^2). \tag{3.7}$$

§ 3.3. Convergence order theorem

Let us formulate and prove a theorem on the order of convergence of the method, following the ideas of the general difference scheme with the effect of heredity [1, 4, 15].

T h e o r e m 3.1. *Suppose that the smoothness conditions of the solution formulated in Lemma* 3.1 *are satisfied, then the error of the method* (2.12) *has order* $h_1^2 + h_2^2 + \Delta^2$.

P r o o f. Let us denote by $\gamma_m = Z^m - U^m$, $m = -M_0, \ldots, M$ vector error. We have $\gamma_m = 0$, $m = -M_0, \ldots, 0$.

For $m = 0, \ldots, M-1$ we have

$$\gamma_{m+1} = S\gamma_m + \Delta\hat{\gamma}_m + \Delta\nu_m, \tag{3.8}$$

where

$$\hat{\gamma}_m = \Phi(t_m, I(\{Z^k\}_m)) - \Phi(t_m, I(\{U^k\}_m)).$$

By virtue of the statement [6, Lemma 4.6],

$$\|(S_y^-)^{-1}(S_x^-)^{-1}\| \leqslant 1.$$

Hence, by the definition (3.6) of the function Φ, the Lipschitz property (1.8) of the function f and the Lipschitz property (2.3) of the operator of piecewise linear interpolation with extrapolation by continuation, we obtain the Lipschitz property of the superposition:

$$\|\hat{\gamma}_m\| \leqslant L \max_{m-M_0 \leqslant k \leqslant m} \{\|\nu_k\|\}, \ L = L_f(L_I + 1). \tag{3.9}$$

Also, we note that $\gamma_0 = 0$, as initial values of the method are equal to exact values.

It follows from (3.8) that

$$\gamma_{m+1} = S^{m+1}\gamma_0 + \Delta \sum_{j=0}^{m} S^{m-j}\hat{\gamma}_j + \Delta \sum_{j=0}^{m} S^{m-j}\nu_j. \tag{3.10}$$

From (3.10), (3.9) and the definition of stability, we have

$$\|\gamma_{m+1}\| \leqslant \hat{S}L\Delta \sum_{j=0}^{m} \max_{j-M_0 \leqslant i \leqslant j} \{\|\gamma_i\|\} + \hat{S}T \max_{0 \leqslant i \leqslant M-1} \{\|\nu_i\|\}. \tag{3.11}$$

We use the notation

$$R = \max_{0 \leqslant i \leqslant M-1} \{\|\nu_i\|\}, \ D = \hat{S}TR. \tag{3.12}$$

Then we can write the estimate (3.11) in the form

$$\|\delta_{m+1}\| \leqslant \hat{S}L\Delta \sum_{j=0}^{m} \max_{j-M_0\leqslant i\leqslant j}\{\|\delta_i\|\} + D. \tag{3.13}$$

Depending on (3.13) and using the mathematical induction, let us prove the estimate

$$\|\delta_m\| \leqslant D(1+\hat{S}L\Delta)^m, \quad n = 1, \ldots, M. \tag{3.14}$$

Induction base. If we put $m = 0$ in (3.13), then

$$\|\delta_1\| \leqslant \hat{S}L\|\delta_0\| + D \leqslant (1+\hat{S}L\Delta)D.$$

Induction step. Let the estimate (3.14) be valid for all indices from 1 to m. We need to show that the estimate is also valid for $m+1$. Fix $j \leqslant m$. Let $i_0 = i_0(j)$ be an index for which $\max_{j-M_0\leqslant i\leqslant j}\{\|\delta_i\|\}$ is obtained. By induction assumption, we gain

$$\max_{j-M_0\leqslant i\leqslant j}\{\|\nu_i\|\} = \|\nu_{i_0}\| \leqslant D(1+\hat{S}L\Delta)^{i_0} \leqslant D(1+\hat{S}L\nu)^j.$$

Thus, the following estimate is also valid

$$\max_{j-M_0\leqslant i\leqslant j}\{\|\nu_i\|\} \leqslant D(1+\hat{S}L\nu)^j.$$

Using the previous inequality and (3.13), we have

$$\|\nu_{m+1}\| \leqslant \hat{S}L\Delta \sum_{j=0}^{m} D(1+\hat{S}L\Delta)^j + D = D(1+\hat{S}L\Delta)^{m+1}.$$

Thus, the estimate (3.14) is proved and this gives

$$\|\nu_m\| \leqslant D\exp(\hat{S}LT). \tag{3.15}$$

Recall the notation of the value D from (3.12) and (3.7), then the inequality

$$D \leqslant C(h_1^2 + h_2^2 + \Delta^2),$$

holds and with the aid of (3.15) the proof is completed. □

§ 4. Numerical experiments

Example 1: Consider the equation

$$\frac{\partial u(x,y,t)}{\partial t} = d_1^+ \frac{\partial_+^{\alpha_1} u(x,y,t)}{\partial x} + d_1^- \frac{\partial_-^{\alpha_1} u(x,y,t)}{\partial x} + d_2^+ \frac{\partial_+^{\alpha_2} u(x,y,t)}{\partial y} + \\ + d_2^- \frac{\partial_-^{\alpha_2} u(x,y,t)}{\partial y} + \int_{-4}^{0} u(x,y,t+s)ds + \bar{f}.$$

Let $\alpha_1 = 1.2,\ \alpha_2 = 1.8$, $\Omega = (0,1)\times(0,1)$ and $T = 8$.

The initial conditions are $u(x,y,t) = e^{-t}x^3(1-x)^3y^3(1-y)^3$, $t \in [-4,0]$, and the boundary conditions are $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$.

Derivative coefficients are determined $d_1^+ = d_1^- = d_2^+ = d_2^- = 1$. The source term is given by

$$\bar{f} = f(x,y,t) + (1-e^4)e^{-t}x^3(1-x)^3y^3(1-y)^3,$$

$$
\begin{aligned}
f(x,y,t) = -e^{-t}\Big[&\big(x^3(1-x)^3y^3(1-y)^3\big) + \\
&+\left(\frac{\Gamma(4)}{\Gamma(2.8)}\big(x^{1.8}+(1-x)^{1.8}\big) - 3\frac{\Gamma(5)}{\Gamma(3.8)}\big(x^{2.8}+(1-x)^{2.8}\big) + \right.\\
&\left.+3\frac{\Gamma(6)}{\Gamma(4.8)}\big(x^{3.8}+(1-x)^{3.8}\big) - \frac{\Gamma(7)}{\Gamma(5.8)}\big(x^{4.8}+(1-x)^{4.8}\big)\right)y^3(1-y)^3 + \\
&+\left(\frac{\Gamma(4)}{\Gamma(2.2)}\big(y^{1.2}+(1-y)^{1.2}\big) - 3\frac{\Gamma(5)}{\Gamma(3.2)}\big(y^{2.2}+(1-y)^{2.2}\big) + \right.\\
&\left.+3\frac{\Gamma(6)}{\Gamma(4.2)}\big(y^{3.2}+(1-y)^{3.2}\big) - \frac{\Gamma(7)}{\Gamma(5.2)}\big(y^{4.2}+(1-y)^{4.2}\big)\right)x^3(1-x)^3\Big].
\end{aligned}
$$

The exact solution is $u(x,y,t) = e^{-t}x^3(1-x)^3y^3(1-y)^3$.

In the following table, we use $N = N_1 = N_2$ to denote the number of spatial partitions in x-direction and y-direction, and use M to denote the number of temporal partition. The maximal error between the true solution and the numerical solution at the last time step is denoted by Error.

Table 1. Error of the numerical solution based on the proposed scheme when $M = N$.

$M = N$	Error
2^3	8.141415×10^{-4}
2^4	9.640892×10^{-5}
2^5	8.843636×10^{-6}
2^6	7.724911×10^{-7}
2^7	7.165794×10^{-8}

Table 2. Error of the numerical solution based on the proposed scheme for different values of M and N.

M	N	Error
2^3	2^4	9.600684×10^{-5}
2^4	2^5	8.835102×10^{-6}
2^5	2^6	7.723120×10^{-7}
2^6	2^7	7.165388×10^{-8}

Table 3. Error of the numerical solution based on the proposed scheme for different values of M and N.

M	N	Error
2^4	2^3	8.159181×10^{-4}
2^5	2^4	9.646250×10^{-5}
2^6	2^5	8.845132×10^{-6}
2^7	2^6	7.725280×10^{-7}

Example 2: Consider the equation

$$
\begin{aligned}
\frac{\partial u(x,y,t)}{\partial t} &= d_1^+\frac{\partial_+^{\alpha_1}u(x,y,t)}{\partial x} + d_1^-\frac{\partial_-^{\alpha_1}u(x,y,t)}{\partial x} + d_2^+\frac{\partial_+^{\alpha_2}u(x,y,t)}{\partial y} + \\
&+ d_2^-\frac{\partial_-^{\alpha_2}u(x,y,t)}{\partial y} + f(x,y,t,u(x,y,t),u(x,y,t-\tau)).
\end{aligned}
$$

Let $\alpha_1 = 1.2,\ \alpha_2 = 1.8$, $\Omega = (0,1) \times (0,1)$, $\tau = 0.5$ and $T = 2$.

The initial condition is $u(x,y,t) = e^{-t}x^3(1-x)^3y^3(1-y)^3$, $t \in [-\tau, 0]$.

The boundary conditions are $u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0$.

Derivative coefficients are determined $d_1^+ = d_1^- = d_2^+ = d_2^- = 1$.

$$
\begin{aligned}
&f(x,y,t,u(x,y,t),u(x,y,t-\tau)) = \\
&= \frac{-e^{-t}}{e^{-t+\tau}x^3(1-x)^3y^3(1-y)^3}\Big[\big(x^3(1-x)^3y^3(1-y)^3\big) + \\
&+ \left(\frac{\Gamma(4)}{\Gamma(2.8)}\big(x^{1.8} + (1-x)^{1.8}\big) - 3\frac{\Gamma(5)}{\Gamma(3.8)}\big(x^{2.8} + (1-x)^{2.8}\big) + \right.\\
&\left. + 3\frac{\Gamma(6)}{\Gamma(4.8)}\big(x^{3.8} + (1-x)^{3.8}\big) - \frac{\Gamma(7)}{\Gamma(5.8)}\big(x^{4.8} + (1-x)^{4.8}\big)\right)y^3(1-y)^3 + \\
&+ \left(\frac{\Gamma(4)}{\Gamma(2.2)}\big(y^{1.2} + (1-y)^{1.2}\big) - 3\frac{\Gamma(5)}{\Gamma(3.2)}\big(y^{2.2} + (1-y)^{2.2}\big) + \right.\\
&+ 3\frac{\Gamma(6)}{\Gamma(4.2)}\big(y^{3.2} + (1-y)^{3.2}\big) - \\
&\left. - \frac{\Gamma(7)}{\Gamma(5.2)}\big(y^{4.2} + (1-y)^{4.2}\big)\right)x^3(1-x)^3\Big]u(x,y,t-\tau).
\end{aligned}
$$

The exact solution is $u(x,y,t) = e^{-t}x^3(1-x)^3y^3(1-y)^3$.

In the following table, we use $N = N_1 = N_2$ to denote the number of spatial partitions in x-direction and y-direction, and use M to denote the number of temporal partition. The maximal error between the true solution and the numerical solution at the last time step is denoted by Error.

Table 4. Error of the numerical solution based on the proposed scheme when $M = N$.

$M = N$	Error
2^3	1.897697×10^{-5}
2^4	2.887081×10^{-6}
2^5	4.644634×10^{-7}
2^6	7.410991×10^{-8}
2^7	1.125744×10^{-8}

Table 5. Error of the numerical solution based on the proposed scheme for different values of M and N.

M	N	Error
2^3	2^4	1.495670×10^{-5}
2^4	2^5	2.161587×10^{-6}
2^5	2^6	3.234001×10^{-7}
2^6	2^7	6.216790×10^{-8}

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Table 6. Error of the numerical solution based on the proposed scheme for different values of M and N.

M	N	Error
2^4	2^3	1.210027×10^{-6}
2^5	2^4	2.692928×10^{-7}
2^6	2^5	7.604774×10^{-8}
2^7	2^6	1.003453×10^{-8}

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М. Ибрагим, В. Г. Пименов

Схема Кранка–Никольсон для дробного двумерного по пространству уравнения диффузии с функциональным запаздыванием

Ключевые слова: дробное уравнение диффузии, две пространственные координаты, функциональное запаздывание, аппроксимация Грюнвальда–Летникова, метод Кранка–Никольсон, факторизация, порядок сходимости.

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Рассматривается двумерное по пространству дробное уравнение диффузии с функциональным запаздыванием общего вида. Для этой задачи конструируется метод Кранка–Никольсон, основанный на сдвинутых формулах Грюнвальда–Летникова для аппроксимации дробных производных по каждой пространственной переменной и применении кусочно-линейной интерполяции дискретной предыстории с экстраполяцией продолжением для учета эффекта запаздывания. Для сведения возникающей системы большой размерности к трехдиагональным системам используется схема Дугласа. Исследована невязка метода. Для получения порядка метода, производится сведение к конструкциям общей разностной схемы систем с наследственностью. Доказана теорема о втором порядке сходимости метода по временным и пространственным шагам. Представлены результаты численных экспериментов.

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